# Math 255A Lecture 11 Notes

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## 1 Applications of Baire's Theorem III: The Uniform Boundedness Principle

### 1.1 Equicontinuity

**Definition 1.1.** A subset $M$ of a locally convex space $V$ is **bounded** if every continuous seminorm $p$ is bounded on $M$: $\sup_{x \in M} p(x) \leq C < \infty$.

When $V_1, V_2$ are locally convex, we let $\mathcal{L}(V_1, V_2)$ be the space of all linear continuous maps $V_1 \to V_2$.

**Definition 1.2.** We say that $\Phi \subseteq \mathcal{L}(V_1, V_2)$ is **equicontinuous** if for every neighborhood $U_2$ of 0 in $V_2$, there is a neighborhood $U_1$ of 0 in $V_1$ such that $x \in U_1 \implies Tx \in U_2$ for every $T \in \Phi$.

If $p_j$ is a continuous seminorm on $V_j$ $(j = 1, 2)$ that $U_j = \{x \in V_j : p_j(x) < 1\}$, then the equicontinuity of $\Phi$ means that $p_1(x) < 1 \implies p_2(Tx) < 1$ for all $T \in \Phi$. This implies that $p_2(Tx) \leq p_1(x)$ for all $x \in V_1$ and $T \in \Phi$. We get that $\Phi \subseteq \mathcal{L}(V_1, V_2)$ is equicontinuous if and only if there exist a continuous seminorm $p_1, p_2$ on $V_1, V_2$ such that

$$p_2(Tx) \leq p_1(x)$$

for all $x \in V_1$ and $T \in \Phi$.

**Remark 1.1.** If $V_1, V_2$ are normed spaces, then $\Phi \subseteq \mathcal{L}(V_1, V_2)$ is equicontinuous means that there exists $C > 0$ such that $\|Tx\|_{V_1} \leq C\|x\|_{V_1}$ for all $x \in V_1$ and $T \in \Phi$. That is, $\|T\|_{\mathcal{L}(V_1, V_2)} \leq C$ for every $T \in \Phi$.

### 1.2 Proof of the uniform boundedness principle

**Theorem 1.1** (Banach-Steinhaus, uniform boundedness principle)**.** *Let $F$ be a Fréchet space, and let $V$ be a locally convex space. If $\Phi \subseteq \mathcal{L}(F, V)$ is such that for each $x \in F$ the set $\{Tx : T \in \Phi\} \subseteq V$ is bounded, then $\Phi$ is equicontinuous. On the other hand, if $\Phi$ is not equicontinuous, then the set of all $x \in F$ such that $\{Tx : T \in \Phi\}$ is bounded is a set of the first category.*

*Proof.* Let $U$ be an open, convex, balanced neighborhood of 0 in $V$, and consider the set $A = \{x \in F : Tx \in \overline{U} \ \ \forall T \in \Phi\} = \bigcap_{T \in \Phi} T^{-1}(\overline{U})$. $A$ is an intersection of closed sets, so it is closed. $A$ is convex as the intersection of convex sets. Also, $A$ is symmetric. Distinguish between two different cases:

1. $A$ has an interior point for any choice of $U$: Then there exists $x_0 \in F$ and a convex, symmetric neighborhood of 0 in $F$ (call it $V$) such that $\{x_0\} + V \subseteq A$. Since $V$ is balanced, $\{-x_0\} + V \subseteq A$, and the convexity of $V$ gives
$$V = \frac{1}{2}(\{x_0\} + V) + \frac{1}{2}(\{-x_0\} + V) \subseteq A.$$
We get that $V \subseteq \bigcap_{T \in \Phi} T^{-1}(\overline{U})$, so $T(V) \subseteq \overline{U}$ for all $T \in \Phi$. So $\Phi$ is equicontinuous.

2. There exists a neighborhood $U$ such that $A = \bigcap_{T \in \Phi} T^{-1}(\overline{U})$ has empty interior. Then $\bigcup_{n=1}^{\infty} nA \subseteq F$ is of the first category, and we claim that it contains the set $\{x \in F : \{Tx : T \in \Phi\} \text{ is bounded}\}$. Take a continuous seminorm $p$ on $V$ such that $\{y : p(y) < 1\} \subseteq U$. Then, since $p(Tx) \leq C$ for all $T \in \Phi$, there exists some $n \in \mathbb{N}$ such that $p(Tx/n) < 1$ for all $T \in \Phi$. So $T(x/n) \in U$ for all $T \in \Phi$, and so $x/n \in A$, which makes $x \in nA$.

To summarize, if $\{Tx : T \in \Phi\}$ is bounded for all $x \in F$, then we are necessarily in case 1 by the open mapping (aka Baire's) theorem. If $\Phi$ is not equicontinuous, we are in case 2, and the set $\{x \in F : \{Tx : T \in \Phi\} \text{ is bounded}\}$ is of the first category in $F$. $\square$

## 1.3 Applications of the uniform boundedness principle

**Corollary 1.1.** *Let $F$ be a Fréchet space, and let $V$ be locally convex and metrizable.*[1] *Let $T_j \in \mathcal{L}(F, V)$ be such that for all $x \in F$, the sequence $(T_j x)$ converges in $V$. Let $Tx = \lim_{j \to \infty} T_j x$. Then $T \in \mathcal{L}(F, V)$.*

*Proof.* Linearity is preserved under limits, so $T$ is linear. For any continuous seminorm $p$ on $V$ and for all $x \in F$, $p(T_j x) \leq C(x)$ for all $j$. By the Banach-Steinhaus theorem, $(T_j)$ is equicontinuous. That is, for every continuous seminorm $p_2$ on $V$, there exists a continuous seminorm $p_1$ on $F$ such that $p_2(T_j x) \leq p_1(x)$ for all $x \in F$ and for all $j$. If we let $j \to \infty$, we get $p_2(Tx) \leq p_1(x)$, so $T \in \mathcal{L}(F, V)$. $\square$

Let $f \in C(\mathbb{R})$ be $2\pi$-periodic. Associated to $f$ is its Fourier series $\sum_{-\infty}^{\infty} c_n(f) e^{inx}$, where
$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$
are the Fourier coefficients. Let $S_N(f, x) = \sum_{-N}^{N} c_n(f) e^{inx}$. Next time, we will show that for all $2\pi$-periodic $f \in C(\mathbb{R})$ outside of a set of the first category, $(S_N(f, x))_{N=1}^{\infty}$ is unbounded for all $x \in \mathbb{Q}$.

[1] The metrizability of $V$ is not actually necessary in this result.